HOW MUCH STATE ASSIGNMENTS CAN DIFFER

TODD A. BRUN

Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540
Phone: 609-734-8335, FAX: 609-951-4489, email: tbrun@ias.edu

The state that an observer attributes to a quantum system depends on the information available to that observer. If two or more observers have different information about a single system, they will in general assign different states. Is there any restriction on what states can be assigned, given reasonable assumptions about how the observers use their information? We derive necessary and sufficient conditions for a group of general density matrices to characterize what different people may know about one and the same physical system. These conditions are summarized by a single criterion, which we term *compatibility*.

1 Observers with differing information

Suppose there is a physical system $\mathcal{S}$ with an associated Hilbert space $\mathcal{H}_S$ of (finite) dimension D, and two observers Alice and Bob each acquire information about $\mathcal{S}$ by some means. Based on this information, they describe $\mathcal{S}$ by two *states*: ρ_A and ρ_B, respectively. If the information that Alice and Bob acquire is not the same, then in general $\rho_A \neq \rho_B$.

The question we address in this paper is this: is there any restriction on the possible assignments ρ_A and ρ_B that can be made? We argue that there is; and we term two states ρ_A and ρ_B which could represent two descriptions of the same physical system *compatible*[1].

For the purposes of this paper, we assume that all of the information acquired by Alice and Bob is both *accurate* and *reliable*. By *accurate* we mean the usual: that any measurements were performed and recorded correctly, and that no one deliberately lied to either of them. By *reliable* we mean something rather more subtle: that the information acquired by Alice and Bob has not been rendered incorrect by disturbances to $\mathcal{S}$ unknown to them.

Classically this is a rather straightforward assumption, but quantum mechanically it is not, largely due to the disturbing effects of measurement. If Bob, for instance, were to measure $\mathcal{S}$, that measurement would disturb the system, potentially introducing errors into Alice's state assignment. By assuming that the information of both observers is *reliable*, we are explicitly ruling out such disturbances. What this means technically will become clear below.

2 Compatibility

Intuitively, what should we expect from a criterion for compatibility? First, if ρ_A and ρ_B are both *pure states*, then they should only be compatible if they are identical:

$$\rho_A = \rho_B = |\psi\rangle\langle\psi| \tag{1}$$

for some $|\psi\rangle$. This seems natural, because pure states represent states of *maximal knowledge* (or minimal ignorance).

Rudolph Peierls[2] suggested two criteria for compatibility in the general case:

$$\begin{aligned} \mathrm{PI}: &\ [\rho_A, \rho_B] = 0 \ , \\ \mathrm{PII}: &\ \rho_A \rho_B \neq 0 \ . \end{aligned}$$

These criteria together rule out differing pure state assignments. Criterion PII seems very natural: it is just the statement that ρ_A and ρ_B are not orthogonal, and hence not contradictory. PI is somewhat less obvious. It seems to be reasoned by analogy with compatible observables, in which $[\hat{A}, \hat{B}] = 0$ implies that the observables A and B can be measured simultaneously.

Unfortunately, in the current case it is much too restrictive. Consider the following example:

$$\begin{aligned} \rho_A &= |\psi\rangle\langle\psi| \\ \rho_B &= p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi| \ , \end{aligned} \tag{2}$$

where $\langle\psi|\phi\rangle \neq 0$. This pair of assignments fails to satisfy PI; but it could easily arise from two observers with different information. For instance, Bob may believe that $\mathcal{S}$ could have been prepared in either state $|\psi\rangle$ or $|\phi\rangle$, while Alice has additional information which rules out the latter.

Let's consider instead one of the following two (equivalent) criteria: ρ_A and ρ_B are compatible if and only if there exist decompositions of ρ_A and ρ_B

$$\begin{aligned} \rho_A &= p_0|\chi\rangle\langle\chi| + \sum_{i>0} p_i |\psi_i\rangle\langle\psi_i| \ , \\ \rho_B &= q_0|\chi\rangle\langle\chi| + \sum_{j>0} q_j |\phi_j\rangle\langle\phi_j| \ , \end{aligned} \tag{3}$$

which share a state $|\chi\rangle$ in common, such that $p_0, q_0 > 0$; or equivalently, if and only if the intersection of their supports is nontrivial,

$$S[\rho_A] \bigcap S[\rho_B] \neq 0 \ , \tag{4}$$

where $S[\rho]$ is the space spanned by the eigenvectors of ρ with nonzero eigenvalues. Note that this definition of compatibility extends straightforwardly to any number of observers, Alice, Bob, Cara, etc. This criterion implies PII, and also implies that pure state assignments must be identical. We show below that this criterion is both necessary and sufficient.

3 Necessity

Assuming Alice and Bob are rational, if they pool their information they should agree on a joint state description ρ_J. Furthermore, since it was assumed that their information was reliable, any measurement outcome to which either of them initially assigned zero probability must *still* have zero probability in the new state ρ_J.

This means that the *null space* of the new state $N[\rho_J]$ must *include* the null spaces of the two states ρ_A and ρ_B:

$$
\begin{aligned}
N[\rho_A] &\subseteq N[\rho_J] \ , \\
N[\rho_B] &\subseteq N[\rho_J] \ ,
\end{aligned}
\tag{5}
$$

(where $N[\rho]$ is the space spanned by the eigenvectors of ρ with vanishing eigenvalues). This implies that $N[\rho_J]$ contains the span of $N[\rho_A]$ and $N[\rho_B]$, which implies in turn that

$$
S[\rho_J] \subseteq S[\rho_A] \bigcap S[\rho_B] \ . \tag{6}
$$

In order for such a joint state assignment to exist, therefore, the intersection of the supports of ρ_A and ρ_B must be nontrivial.

4 Sufficiency

Obviously, if one is just given two state assignments $\rho_{A,B}$ it is impossible to know if they are intended to apply to the same system or not. So by sufficiency what we mean is that if the assignments satisfy the compatibility criterion, then they *could* be different descriptions of the same system based on different information.

We prove this by construction. Suppose ρ_A and ρ_B have decompositions (3). These state assignments could have arisen in the following way. Suppose that there are two ancillary systems $\mathcal{A}$ and $\mathcal{B}$ in addition to $\mathcal{S}$, and that both Alice and Bob know that the initial state was

$$
\begin{aligned}
|\Psi\rangle = \frac{1}{N}\Bigg(& |0\rangle_{\mathcal{A}}|0\rangle_{\mathcal{B}}|\chi\rangle_{\mathcal{S}} + \sum_{i>0}\sqrt{\frac{p_i}{p_0}}|0\rangle_{\mathcal{A}}|i\rangle_{\mathcal{B}}|\psi_i\rangle_{\mathcal{S}} \\
& + \sum_{j>0}\sqrt{\frac{q_j}{q_0}}|j\rangle_{\mathcal{A}}|0\rangle_{\mathcal{B}}|\phi_j\rangle_{\mathcal{S}} \Bigg) \ ,
\end{aligned}
\tag{7}
$$

where N is a normalization factor, $|0\rangle_{\mathcal{A}}$ and $|j\rangle_{\mathcal{A}}$ are all mutually orthogonal, and similarly for $|0\rangle_{\mathcal{B}}$ and $|i\rangle_{\mathcal{B}}$. Alice then measures subsystem $\mathcal{A}$ and Bob measures $\mathcal{B}$, both getting result 0, but they do not share their results. It is clear that in this case Alice and Bob will make the state assignments ρ_A and ρ_B given in (3). If they were to pool their information, they would both arrive at the joint state assignment $|\chi\rangle$. So any state assignments which satisfy the compatibility criterion *could* arise from observers with different information about the same physical system $\mathcal{S}$.

5 Further questions

Both the problem and the solution are easy to state, but lead one into subtle and interesting questions about how state assignments are made. For example:

1. What restrictions are there on how Alice and Bob acquire their information, if we want it to be *accurate* and *reliable*? How does the case where they perform measurements differ from the case where they get information from a knowledgeable third party?

2. If Alice and Bob pool their information, how do they form a joint state assignment[3]? Rather than giving an all or nothing criterion, is it possible to quantify a *degree* of compatibility between two state assignments[4]?

3. Are there other reasonable notions of compatibility? And if so, do they lead to well-defined compatibility criteria[5]?

All of these questions have been examined to some extent, but much remains open. This just shows how far there is to go in completely understanding state assignment in quantum mechanics.

Acknowledgments

First and foremost, I would like to acknowledge my collaborators Jerry Finkelstein and David Mermin; all technical results of this paper were developed jointly with them. I thank Robin Blume-Kohout, Carl Caves, Jennifer Dodd, Chris Fuchs, David Poulin and Rüdiger Schack for stimulating conversations and feedback, and the organizers of QCMC 2002. This research was supported in part by the Martin A. and Helen Chooljian Membership in Natural Sciences, and by DOE Grant No. DE-FG02-90ER40542.

References

1. Todd A. Brun, J. Finkelstein and N. David Mermin, Phys. Rev. A 65, 032315 (2002).
2. R.E. Peierls, Physics World, 19-21 (January 1991).
3. Kurt Jacobs, quant-ph/0201096.
4. One such measure was suggested by David Poulin and Robin Blume-Kohout in quant-ph/0205033. Another measure, suggested by Alexei Kitaev (private communication), is the maximum trace of a positive operator $\hat{R}$ such that $\rho_A - \hat{R}$ and $\rho_B - \hat{R}$ are both positive.
5. Carlton M. Caves, Christopher A. Fuchs and Rüdiger Schack, quant-ph/0206110.